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A superspace path integral proof of the Gauss-Bonnet-Chern theorem

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Abstract. *A rigorous theory of integration in the space of paths in super space is developed, by extending Berezin's method of integration to spaces of anticommuting variables with an uncountably high dimension. A Feynmam-Kac-Ito formula for the heat kernel of a wide class of superspace differential operators is established. This formula is then used to make rigorous the supersymmetric proofs of the Gauss-Bonnet-Chern theorem [1, 2].*

1. INTRODUCTION

The purpose of this paper is to develop an analytic theory of combined bosonic and fermionic path integrals in curved space, so that recent simple proofs of the Gauss-Bonnet-Chern theorem by Alvarez-Gaume [1] and by Friedan and Windey [2], which use such path integrals, can be made rigorous.

The Gauss-Bonnet-Chern theorem relates the Euler number of a Riemannian manifold to the integral of its top Euler class. If M is a compact orientable Riemannian manifold of even dimension $n = 2k$, with metric g, the theorem states that

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(1.1) $$\chi = \int_M \mathbf{E}$$

where χ is the Euler number of M and E is a representative of its to Euler class. If $\mathrm{d}V$ denotes the volume form on M corresponding to the metric g one may use

(1.2) $$\mathbf{E} = E\,\mathrm{d}V$$

with the scalar function E given by

(1.3) $$E = (1/k!)(-1)^k(8\pi)^{-k} \times \sum_{p,q\in \mathrm{perm}(1,..,n)} (-1)^p(-1)^q R^{p(1)p(2)}_{q(1)q(2)} \dots R^{p(n-1)p(n)}_{q(n-1)q(n)}.$$

This theorem was originally proved by Chern in 1944 [3], and is also one of many results contained in the Atiyah-Singer index theorem [4]. Recently very simple, but not completely rigorous, proofs of the Ativah-Singer index theorem have been given by Alvarez-Gaume [1] and by Friedan and Windey [2], making use of Witten's observations [5] on properties of supersymmetric quantum mechanics. These proofs, which will be referred to as the supersymmetric proofs, have two essential ingredients. First there is McKean and Singers' expression of the Euler number in terms of the supertrace of the operator $\exp(-\Delta t)$ (Δ being the Laplacian on forms on the manifold) [6],

(1.4) $$\chi = Str\,\exp(-\Delta t).$$

(This expression was rediscovered by Witten in the language of supersymmetry). Secondly there is the use of path integral methods (including fermionic path integrals) to calculate this supertrace.

McKean and Singer's formula is perfectly rigorous; the stage at which lack of complete rigour comes in to the supersymmetric proofs is in the path integrals, particularly in the use of fermionic path integrals which are integrals in spaces of paths in some anticommuting space. The purpose of this paper is to develop a theory of combined bosonic and fermionic path integration which makes the methods used in the proof valid in as straightforward way as possible, so that the clarity and simplicity of the proofs are not lost.

As with other heat kernel proofs of the index theorem [7 - 13], the strategy of the supersymmetric proofs is to prove a stronger local version of the Gauss-Bonnet-Chern theorem first conjectured by McKean and Singer [6]. This states that

(1.5) $$\lim_{t\to 0} str(\exp(-\Delta t))(x,x) = E(x)$$

at each point x of M. (Here *str* denotes a $2^n \times 2^n$ matrix supertrace, whereas *Str* in equation (1.4) denotes a full supertrace in both the matrix and operator sense. Both these supertraces are fully defined in section 2). Standard cutting and pasting arguments mean that one can work in R^n to evaluate the density $\exp(-\Delta t)(x, x)$ in the limit $t \to 0$.

The rigorous version of the supersymmetric proofs which is presented in this paper is developed in three stages. In section 2, beginning with the natural correspondence between forms on R^n and functions on (n, n)-dimensional superspace,

$$(1.6) \qquad \sum_{1=i_1<\dots<i_p}^{n} a_{i_1\dots i_p}(x)\, dx^{i_1} \dots dx^{i_p} \longleftrightarrow \sum_{1=i_1<\dots<i_p}^{n} a_{i_1\dots i_p}(x)\, \theta^{i_1} \dots \theta^{i_p}.$$

it is shown how the Laplacian $\Delta = (d + \delta)^2$ can be expressed as a differential operator on the space of smooth functions on the superspace. This leads to a superspace expression for the density *str* $\exp(-\Delta t)\,(x, x)$.

In section 3 techniques of path integration in spaces of anticommuting variables, developed by the author in a previous paper [14], are combined with conventional path integration methods to give a formula for the heat kernel of the superspace version of the Laplacian. The proof of the Gauss-Bonnet-Chern theorem is completed in section 4 by using the path integral techniques to evaluate the limit as $t \to 0$ of the matrix supertrace of the superspace heat kernel. The method both picks out the constant term, and shows that the remaining terms tend to zero with t, in a very simple way.

Methods involving anticommuting variables have mostly been used for formal heuristics, rather than solid analysis. The results of this paper show that anticommuting variables can be used for rigorous analysis; moreover the process of making the physicist's heuristic approach rigorous does not destroy the beautiful simplicity of the physicist's argument.

There are of course many other proofs of the Gauss Bonnet Chern theorem, and indeed the full Atiyah-Singer index theorem. Of some relevance to the methods of this paper are Bismut's probabilistic proof [10], and two different analytic proofs, by Getzler [11] and by Cycon, Froese, Kirsch and Simon [12]. There is also the proof by Lott [13], which uses the approach to Fermionic path integration developed by Osterwalder and Schrader [15]. This approach is different from that of this paper, which takes as literally as possible the idea introduced by physicists of a measure (of Berezin type) in the space of paths in superspace. The reason for burdening the literature with yet another proof of the Gauss Bonnet Chern theorem is to demonstrate that this kind of path integration allows one to make rigorous the formal manipulations of path integrals which are so intuitively appealing.

2. A SUPERSPACE EXPRESSION FOR THE EULER NUMBER

As before, let M be an n-dimensional compact orientable Riemannian manifold, with metric g. In this section the density of the Euler number of M will be expressed in terms of the kernel of a differential operator on (n, n) dimensional superspace.

2.1. Preliminary definitions

A brief summary of the techniques of analysis of functions of even and odd variables will be given; further details may be found in [16, 17, 18]. The concrete approach to functions of anticommuting variables will be used here, in which functions between actual spaces are considered. There is an equivalent more abstract approach to such functions (developed in a global context) by Kostant [19] and by Berezin and Leites [20]. Although the abstract approach is more elegant in that it does not require the introduction of an auxilliary Grassmann algebra, the language of the concrete approach used in this paper is more appropriate for path integration. It allows a more direct generalisation of conventional bosonic path integration, and does not hide the conceptual simplicity of the path integral method in complicated formal constructions. Also, it is close to the physicists' method which inspires the approach.

For each positive integer L, B_L will denote the Grassmann algebra $\Lambda(R^L)$ while C_L will denote the Grassmann algebra $\Lambda(C^L)$.(B_L is regarded as embedded in C_L via a basis of generators). $B_{L,1}$ will denote the odd part of B_L. Also, for each pair of positive integers r, s $R_L^{r,s}$ will denote the Cartesian product $R^r \times (B_{L,1})^s$; a typical element of this space will be denoted $(x^1, \ldots, x^r; \theta^1, \ldots \theta^s)$ or (x, θ). The topology used on B_L, $B_{L,1}$ and $R_L^{r,s}$ will simply be the usual topology of a finite dimensional vector space. For analytic purposes it is useful to use an l^1 norm on B_L (with respect to a basis of generators); B_L then becomes a Banach algebra. ($R_L^{r,s}$ is more restricted than the flat superspace $B_L^{r,s}$ of reference 18, but sufficient for this paper.)

It is useful to introduce a notation for a commonly occurring set of multi-indices. The set of multi-indices $\mu = \mu^1 \ldots \mu^k$, with $1 \leqslant \mu^1 < \ldots < \mu^k \leqslant s$ (including the empty index, denoted ϕ), will be denoted M_s. Products such as $\theta^{\mu_1} \ldots \theta^{\mu_k}$ will be abbreviated as θ^μ. Multi-indices will be summed over M_s for the appropriate value of s. Single indices will be denoted by latin letters: repeated latin indices will be summed over their range, according to the usual Einstein summation convention.

Various spaces of functions on $R_L^{r,s}$ will now be defined. A function $f : R_L^{r,s} \to B_L$ will be said to be C^∞ if, for each μ in M_s, there exists a C^∞ function $f_\mu : R^r \to R$ such that

(2.1.1) $$f(x, \theta) = f_\mu(x)\theta^\mu.$$

($C^{\infty\prime}$ functions are equivalent to the H^∞ functions of reference 18 restricted to $R_L^{r,s}$). The space of all such $C^{\infty\prime}$ functions will be denoted $C^{\infty\prime}(R_L^{r,s}, R)$. In a similar way $C^\infty(R_L^{r,s}, C)$ is defined to be the space of all functions of $R_L^{r,s}$ into C_L expressible in the form (2.1.1) with the coefficient functions f_μ required to be C^∞ functions of R^r into C. More generally, if A is a finite-dimensional algebra over the reals, $C^\infty(R_L^{r,s}, A)$ denotes the space of functions f of $R_L^{r,s}$ into $A \times B_L$ satisfying

(2.1.2) $$f(x, \theta) = f_\mu(x)\,\theta^\mu$$

where the coefficient functions f_μ are C^∞ functions of R^r into A.

Even derivatives ∂_i and odd derivatives δ_i of a $C^{\infty\prime}$ function are defined by

$$\partial_i f_\mu(x^1 \ldots, x^r; \theta^1 \ldots \theta^s) = \partial_i f(x^1 \ldots, x^r)\theta^\mu$$

for $i = 1 \ldots, r$

and $\delta_j f_\mu(x^1 \ldots, x^r)\theta^\mu$ (no summation)

(2.1.3)
$$= -(-1)^p f_\mu(x^1 \ldots, x^r)\theta^{\mu_1} \ldots \theta^{\mu_{p-1}}\theta^{\mu_{p+1}} \ldots \theta^{\mu_k} \quad \text{if} \quad j = \mu^p$$
$$= 0 \quad \text{otherwise for} \quad j = 1 \ldots, s.$$

Other spaces of functions (such as $L^{2\prime}(R_L^{r,s}, C)$ and $C_0^{\infty\prime}(R_L^{r,s}, R)$ will be defined in a similar way by imposing the appropriate restrictions on the coefficient functions f_μ. A norm $\|\ \|$ will be defined on $L^{2\prime}(R_L^{r,s}, C)$ by setting

(2.1.5) $$\|f\| = \sum_{\mu \in M_s} \|f_\mu\|.$$

With this norm $L^{2\prime}(R_L^{r,s}, C)$ becomes a Hilbert space. Also $L^{2\prime}(R_L^{r,s}, C)$ is the closure of $C_0^{\infty\prime}(R_L^{r,s}, C)$ with $L^{2\prime}$ norm.

2.2. Integral kernels on superspace

Suppose that $K : L^{2\prime}(R_L^{r,s}, C) \to L^{2\prime}(R_L^{r,s}, C)$. Then, if there exists a function

$$K : B_L^{r,s} \times B_L^{r,s} \to C_L$$

polynomial in anticommuting arguments such that, for all f in $L^{2\prime}(R_L^{r,s})$ and all (x, θ) in $R_L^{r,s}$, the integral $\int d^n y\, d^n\phi K(x, y, \theta, \phi) f(y, \phi)$ exists and satisfies

(2.2.1) $$K(f)(x, \theta) = \int d^n y\, d^n \phi\, K(x, y, \theta, \phi) f(y, \phi),$$

the function K is said to be the integral kernel of the operator K. (The integration over the anticommuting variables ϕ_j in (2.2.1) is carried out according to the usual Berezin prescription [21]). Kernel of operators on other spaces of functions of $R_L^{r,s}$ may be defined similarly when appropriate.

It is useful to notice that, if $r = 0$ (so that a purely odd superspace is being considered) and s is even, then $\delta(\theta, \phi)$, the kernel of the identity operator on $L^{2'}(R_L^{o,s})$ may be expressed as

(2.2.2) $$\delta(\theta, \phi) = \int d^s k \exp - ik_j(\theta^j - \phi^j)$$

while the operator $a_\mu^\upsilon \theta^\mu \delta_\upsilon$ has kernel

(2.2.3) $$a_\mu^\upsilon \theta^\mu \delta_\upsilon(\theta, \phi) = \int d^s k\, a_\mu^\upsilon \theta^\mu k_\upsilon \exp - ik_j(\theta^j - \phi^j).$$

(Here δ_υ denotes the multiderivative $\delta_{\upsilon_1} \ldots \delta_{\upsilon_k}$).

More generally, there is a natural identification

$$i : L^{2'}(B_L^{r,s}) \to (L^2(\mathbb{R}^r))^{2^s}$$

with

(2.2.4) $$i(f) = (f_\mu \mid \mu \in M_s)$$

if

$$f(x, \theta) = f_\mu \theta^\mu .$$

An operator K on $L^{2'}(B_L^{r,s})$ has an integral superkernel if and only if the corresponding operator $i^{-1} \circ K \circ i$ on $(L^2(\mathbb{R}^r))^{2s}$ has a (matrix valued) integral kernel $K(x, y)$. That is, for each x, y in $\mathbb{R}^r \times \mathbb{R}^r$, $K(x, y)$ is a $2^s \times 2^s$ matrix with complex entries $K_\mu^\upsilon(x, y)$ such that, for all x in R^r and (f_μ) in $(L^2(\mathbb{R}^r))^{2s}$,

(2.2.5) $$\int d^r x\, K_\mu^\upsilon(x, y) f_\upsilon(y) = ((i^{-1} \circ K \circ i) f)_\mu(x).$$

The matrix supertrace of $K(x, y)$ is defined by

(2.2.6) $$\operatorname{str}(K(x, y)) = \sum_{\mu \in M_s} (-1)^{|\mu|} K_\mu^\mu ,$$

where $|\mu|$ denotes the number of elements in the multi-index μ.

A key property of the super heat kernel, which may be proved by explicit calculation, is that

$$\int d^n\theta \, K(x, y, \theta, \theta) = \operatorname{str} K(x, y). \tag{2.2.7}$$

2.3. A superspace expression for the Euler number density of M

The Euler number density of the n-dimensional manifold M with Riemannian metric g will now be expressed in terms of the superkernel of an operator on $C^{\infty'}(B_L^{n,n})$. (Cutting and pasting arguments mean that it is not necessary to use any global supermanifold constructions). There is a natural identification j of $C^{\infty'}(R_L^{n,n})$ with $\Omega(R^n)$ (the space of smooth forms on R^n) defined by

$$\begin{aligned} j(f) &= f_\mu \, dx^\mu \\ f(x, \theta) &= f_\mu \theta^\mu . \end{aligned} \tag{2.3.1}$$

(Here the multi-indices μ are summed over the space M_n). There is a similar identification of $\bar{\Omega}(R^n)$ (the L^2 closure of $\Omega(R^n)$) and $L^{2'}(R_L^{n,n}, C)$. Clearly an operator K on $\bar{\Omega}(R^n)$ has an integral kernel if and only if the corresponding operator $j^{-1} \circ K \circ j$ on $L^{2'}(R_L^{n,n})$ has an integral superkernel. The operator of importance in section 4 is the Laplacian Δ' on $\Omega(R^n)$ corresponding to a metric g'. Under the identification j, Δ' is replaced by the operator

$$\begin{aligned} \Delta'' &= j^{-1} \circ \Delta \circ j \\ &= -g^{ij\prime}(\partial_1 + \Gamma^{1\prime}_{ik}\theta^k \delta_1)(\partial_j + \Gamma^{n\prime}_{jm}\theta^m \delta_n) \\ &\quad + R^{k\prime}_j \theta^j \delta_k + (1/2) R^{ij\prime}_{k1} \theta^k \theta^1 \delta_i \delta_j \end{aligned} \tag{2.3.2}$$

acting on $C^{\infty'}(R_L^{r,s})$, (Recall that ∂_i denotes differentiation with respect to x^i and δ_i denotes differentiation with respect to θ^i, as in equation (1.3)).

Let Δ denote the Laplacian on $\Omega(M)$ corresponding to the metric g. For the proof of the Gauss Bonnet Chern theorem given in the next section, it is necessary to prove the existence of the limit as t tends to zero of $\operatorname{str}\exp(-t\Delta)(x, x)$ for each point x of M, and to evaluate this limit. Standard cutting and pasting arguments (described in the book of Cycon, Froese, Kirsch and Simon [12]) show that it is sufficient to work on R^n in the following manner. Suppose that the point x in M lies in a coordinate chart U, with corresponding normal coordinate mapping $\Psi : U \to R^n$ such that $\Psi(x) = 0$, and the closure of $\Psi(U)$ is compact. Then a metric g' on R^n is chosen such that $g' = \Psi^{-1*}(g)$ on $\Psi(U)$ and g' is the Euclidean metric outside a compact set V containing $\Psi(U)$. It can then be shown that, if Δ' is the Laplacian on $\Omega(R^n)$ corresponding to the metric g', then

(2.3.3) $$|\exp - \Delta t(x, x) - \exp - \Delta' t(0, 0)| = 0(\exp - c/t)$$

where c is a positive constant. Hence, when evaluating χ using equation (1.5) one may replace str exp $- t\Delta(x, x)$ by str $\exp(-\Delta' t)(0, 0)$; combining this with (2.2.7) one obtains

(2.3.4) $$\lim_{t\to 0} \text{str} \exp - t\Delta(x, x) = \lim_{t\to 0} \int d^n\theta \exp(-(\Delta'')t)(0, 0, \theta, \theta).$$

Thus one has an expression for the Euler number density of M in terms of the super kernel of an operator on $L^{2'}(R_L^{n,n})$. In the following sections path integral techniques will, be developed to evaluate this super kernel, and hence prove the Gauss-Bonnet-Chern formula.

3. SUPERSPACE PATH INTEGRATION

In this section a measure on the space of paths in superspace is developed by combining conventional bosonic path integrals with the fermionic path integrals previously introduced by the author [14]. This measure is then used to derive a Feynman-Kac-Ito formula for the kernel of a wide class of differential operators on $C^{\infty'}(R_L^{r,s})$. (In section 4 this formula will be useto to prove the Gauss-Bonnet-Chern theorem). In 3.1 a general formalism for integration on infinite dimensional superspace is set up. Concepts of super measure and super random variable are described. In 3.2 a particular super measure is defined on the space of paths in superspace, while 3.3 contains the path integral expression for the super kernel of the evolution operator of a wide class of Hamiltonians.

3.1. Measures in infinite dimensional superspace

Let A be a set, and L, r and s be positive integers with $s < L$. Let C_L denote the complex Grassmann algebra with L anticommuting generators $b_1, \ldots, b_L$, and let B_L denote the real Grassmann algebra with the same generators. For each $t \in A$ let ${}_tB_{L,1}$ denote the odd part of the real Grassmann algebra with L generators ${}_tb_1, \ldots, {}_tb_L$, and let

(3.1.1) $$B_L^{(r,s)A} = (R^r)^A \times \prod_{t\in A} {}_tB_{L,1}.$$

(The more natural set to use would be $(R_L^{r,s})^A$, the Cartesian product of copies of $R_L^{r,s}$; it is not used because many of the functions required on this space would be zero for the quite trivial reason that any product of $L + 1$ odd elements of B_L must be zero). A typical element of $R_L^{(r,s)A}$ will be denoted $(x^i(t), \theta^j(t))$.

The approach taken to measures on $R_L^{(r,s)A}$ will be a combination of conventional measures on $(R^r)^A$ (in the Bochner approach, where finite dimensional marginals are the primary objects [22]) with the Berezin type measures on $R_L^{(o,s)A}$ introduced by the author in [14]. These bosonic and fermionic type measures are combined to give the following definition of super probability measure on $R_L^{(r,s)A}$.

DEFINITION 3.1.1. An (r, s)-super probability space of weight $w \in C_L$ consists of

a) a set A

b) for each finite subset $J = t_1, \ldots, t_k$ of A a function $f_\sigma \in L^{2'}(R_L^{(r,s)J} \cdot C_L)$ such that

(i)
$$\int d^{rk}x \, d^{sk}\theta \, f_J(x \cdot \theta) = w$$

(ii) if $J' = \{t_1, \ldots, t_{k-1}\}$ then

$$f_{J'}(x^1, \theta^1, \ldots, x^{k-1}, \theta^{k-1})$$
$$= \int d^r x^k \, d^s\theta^k f_J(x^1, \theta^1, \ldots, x^k, \theta^k). \tag{3.1.2}$$

∎

Such a super probability space will be denoted $\{R_L^{(r,s)A}, \{f_J\}, d\mu\}$. The f_J will be referred to as the finite distributions of the super measure $d\mu$.

The aim of this approach is to build Kolmogorov-type consistency conditions into the definition and thus be able to integrate certain objects, not quite functions of $R_L^{(r,s)A}$, which will be called super random variables.

DEFINITION 3.1.2. A super random variable g on an (r, s)-super probability space $\{R_L^{(r,s)A}, \{f_J\}, d\mu\}$ consists of

(i) a sequence $J(1), J(2), \ldots$ of finite subset of A

(ii) for each positive integer N a function g_N in $L^{2'}(B_L^{(r,s)J(N)}, C_L)$ such that (if $\#(N)$ denotes the number of elements in the set $J(N)$) the sequence $I_1, I_2, \ldots$ with

$$I_N = \text{def} \int d^{r\#(N)}x \, d^{s\#(N)}\theta \;\; f_{J(N)}(x, \theta) g_N(x, \theta)$$

tends to a limit as N tends to infinity. This limit will be denoted $\int d\mu \, g$.

3.2. A super-measure on paths in superspace

A super measure on paths in the superspace $R_L^{n,2n}$ which is a generalisation of Wiener measure will now be introduced. This particular measure is designed to make the proof of the Gauss-Bonnet-Chern theorem in section 4 of this paper as straightforward as possible. It will be used in 3.3 to give a path integral expression for the super kernel of the operator $\exp - \Delta'' t$ (where Δ'' is the localised superspace Laplacian defined in equation (2.3.2)). Similar measures have been used by Elworthy [23] in the purely commuting case.

Throughout this section, and the rest of this paper, i is an index which runs from i to n.

DEFINITION 3.2.1. (a) Let I be the open interval $(0, t)$, $a, b \in R^n$ and $\alpha, \beta, \kappa \in$ $\in R_L^{o,n}$. Let g' be a metric on R^n equal to the Euclidean metric outside a compact set V containing the origin 0. Also for $0 < s \leqslant t$ let $P_s(x, y)$ be the kernel of the operator $\exp - tH_0$ where H_0 is the scalar Laplacian $\partial_l g^{il} \partial_j$ acting on $C^\infty(R^n)$. The super measure $d\mu\,[a, b, \alpha, \beta, \kappa, t]$ is defined by the finite distribution functions $f_J \in C^{\infty'}(R_L^{(n,2n)J}, C_L)$ where, if $J \subset I$ with $J = t_1, \ldots, t_N$ and $t^1 < \ldots < t^N$,

$$f_J(x^{1i}, \theta^{1i}, \rho_{1i}, \ldots, x^{N1}, \theta^{Ni}, \rho_{Ni})$$

$$(3.2.1) \qquad = P_{t_1}(a, x^1) P_{t_2 - t_1}(x^1, x^2) \ldots P_{t - t_N}(x^N, b)$$

$$x \exp - i\{(\kappa_i(\theta^{1i} - \alpha^i) + \rho_{i1}(\theta^{2i} - \theta^{1i}) + \ldots + \rho_{Ni}(\beta^i - \theta^{Ni})\} \qquad \blacksquare$$

(It may easily be checked, using the semigroup property of $\exp - tH_0$ that the finite distributions f_J satisfy the consistency conditions (i) and (ii) of definition 3.1.1 and thus define a super measure).

The purpose of introducing this measure on the space of paths in superspace is to use it to develop a Feynman-Kac-Ito formula for the kernel of certain operators on $L^{2'}(R_L^{n,n})$. As a first step, the existence of a class of random variables on the super measure space $((R_L^{(n,2n)I}, f_J, d\mu\,[a, b, \alpha, \beta, \kappa, t])$ will be established; random variables of this kind will form the integrand in the Feynman-Kac-Ito formula.

PROPOSITION 3.2.2. *Let t be a positive real number. For each $N = 1, 2, \ldots,$ let*

$$(3.2.2) \qquad J(N) = (rt/2^N \mid r = 1, \ldots, 2^{N-1})$$

Also suppose that, for $i = 1, \ldots, n$ and μ, v in M_n, $h_{i\mu}^v$ and k_μ^v are complex-valued C^∞ functions of R^n with compact support. For $N = 1, 2, \ldots,$ let functions g_N in $C^{\infty'}(R_L^{(n,2n)J(N)}, C_L)$ be defined by

$$g_N(x^{1i}, \theta^{1i}, \rho_{1i}, \ldots, x^{Ni}, \theta^{Ni}, \rho_{Ni})$$

(3.2.3)
$$= \exp\left\{\sum_{r=0}^{2N-1}\left[\frac{1}{2}\,(h^{v}_{i\mu}(x^r)\theta^{r\mu}\rho_{rv} + h^{v}_{i\mu}(x^{r+1})\theta^{r\mu}\rho_{rv})(x^{ri} - x^{r+1i}) + (t/2^N)K^{v}_{\mu}(x^r)\theta^{r\mu}\rho_{rv}\right]\right\}$$

(where, for notational convenience, $x_0 = a$, $x_2 = b$, $\theta_0 = \alpha$, $\theta_2 = \beta$ and $\rho_0 = x$). Then the sequence $g_1, g_2, \ldots$ defines a super random variable on the super probability space $(R_L^{(n,2n)I}, (f_J), \mathrm{d}\mu)$. (This random variable will be denoted

$$\exp\left(\int_0^t h^{v}_{i\mu}(x(u))\theta^{\mu}(u)\rho_v(u)\,\mathrm{d}x^i(u) + \int_0^t k^{v}_{\mu}(x(u))\theta^{\mu}(u)\rho_v(u)\,\mathrm{d}u\right).$$

Outline of proof. For each positive integer N let $\mathrm{d}\mu_N$ denote

$$\prod_{r=1}^{N} \mathrm{d}^n x^r\ \mathrm{d}^n\theta^r\ \mathrm{d}^n\rho_r\ f_{J(N)}(x^{1i}, \theta^{1i}, \rho_{1i}, \ldots, x^{Ni}, \theta^{Ni}, \rho_{Ni}),$$

and let

$$I_N = \int \mathrm{d}\mu_N\ g_N(x^{1i}, \theta^{1i}, \rho_{1i}, \ldots, x^{Ni}, \theta^{Ni}, \rho_{Ni}).$$

Also let $\delta t = t/2^{N+1}$. Then

$$I_{N+1} - I_N = \sum_{p=1}^{2^N} S_{N,P+1} - S_{N,P}$$

where

$$S_{N,P} = \int \mathrm{d}\mu_N$$

$$\times \exp\left(\sum_{r=0}^{2P-1}\left[\frac{1}{2}\,((h^{iv}_{\mu}(x^r)\theta^{r\mu}\rho_{rv} + h^{iv}_{\mu}(x^r)\theta^{r\mu}\rho_{rv})(x^{ri} - x^{r+1})\right.\right.$$

$$+ \delta t\, K_{\upsilon}^{\mu}(x^{r})\theta^{r\mu}\rho_{r\upsilon}\Big]$$

(3.2.4)
$$+\left(\sum_{r=P}^{2N-1}\left[\frac{1}{2}\left((h_{\mu}^{i\upsilon}(x^{2r})\theta^{r\mu}\rho^{r}\upsilon + h_{\mu}^{i\upsilon}(x^{2(r+1)})\theta^{r\mu}\rho^{r}\upsilon)(x^{2si}-x^{2(s+1)i}\right)\right.\right.$$
$$\left.\left. + 2\delta t\, K^{\mu}\upsilon(x^{2s})\theta^{2r\mu}\rho^{2r}\upsilon\right]\right)$$

Because of the compact support of $h_{i\mu}^{\upsilon}$ and K_{μ}^{υ}, the exponent in (3.2.4) is bounded and so the exponential can be expanded in a power series in the exponent. In evaluating $S^{N,P+1} - S^{N,P}$ the contributions from the zeroth order terms in the exponential cancel; it will now be shown that the contribution from the first order terms is bounded by $C P_t(a, b)M$, where

$$M = |\exp - ik_1\alpha^i| \exp\left\{\sum_{\mu\in M_n}(|\kappa_\mu| + |\beta^\mu|)\right\}$$

and C is a positive constant specified below. The first order contribution to $S_{N,P+1} - S_{N,P}$ is

$$\int d\mu_{N+1}\,\Delta K(\theta) + \Delta K(\rho) + \Delta K(x) + \Delta h(\theta) + \Delta h(\rho) + \Delta h(x)$$

where

$$\Delta K(\theta) = \delta t\, K_{\mu}^{\upsilon}(x^{2P})\theta^{2P+1\mu}\rho_{2P+1\upsilon} - K_{\mu\upsilon}(x^{2p})\theta^{2P\mu}\rho_{2P+1\upsilon}$$
$$\Delta K(\rho) = \delta t\, K_{\mu}^{\upsilon}(x^{2P})\theta^{2p\mu}\rho^{2P+1\upsilon} - K_{\mu}{}^{\upsilon}(x^{2P})\theta^{2P\mu}\rho_{2P\upsilon}$$
$$\Delta K(x) = \delta t\, K_{\mu}^{\upsilon}(x^{2P+1})\theta^{2P+1i}\rho_{2P+1i} - K_{\mu}{}^{\upsilon}(x^{2P})\,\theta^{2P+1\mu}\rho_{2P+1\upsilon}$$

$$\Delta h(\theta) = \frac{1}{2}\,(h_{i\mu}^{\upsilon}(x^{2P+1}) + h_{i\mu}^{\upsilon}(x^{2P+2}))(\theta^{2P+1\mu} - \theta^{2P\mu})\rho_{2P+1\upsilon}$$

(3.2.5)
$$\times\,(x^{2P+1i} - x^{2P+2i})$$

$$\Delta h(\rho) = \frac{1}{2}\,(h_{i\mu}^{\upsilon}(x^{2P+1}) + h_{i\mu}^{\upsilon}(x^{2P+2}))\theta^{2P\mu}(\rho_{2P+1\upsilon} - \rho_{2P\upsilon})$$
$$\times\,(x^{2P+1i} - x^{2P+2i})$$

$$\Delta h(x) = \frac{1}{2}\,[(h_{i\mu}^{\upsilon}(x^{2P+2}) - h_{i\mu}^{\upsilon}(x^{2P+1}))(x^{2P+1i} - x^{2Pi})$$
$$- (h_{i\mu}^{\upsilon}(x^{2P+1}) - h_{i\mu}^{\upsilon}(x^{2P}))(x^{2P+2i} - x^{2P+1i})]\theta^{2P\mu}\rho_{2P\upsilon}.$$

Now

$$\int d\mu_{N+1}\,\Delta K(\theta) = \int d\mu_{N+1}\,\Delta K(\rho) = \int d\mu_{N+1}\,\Delta h(\theta)$$

$$= \int d\mu_{N+1}\,\Delta h(\rho) = 0,$$

because of the δ-function nature of the odd part of the measure.

Let

$$K(x) = \sum_{\mu,\upsilon \in M_n} |K_\mu^\upsilon(x)| \,.$$

$$t_1 = 2P\,\delta t \quad \text{and} \quad t_2 = (2^{N+1} - 2P - 1)\delta t.$$

Then

$$\left| \int d\mu_{N+1}\,\Delta K(x) \right|$$

$$\leqslant M\,\delta t\,|\exp(-H_0 t_1)(\exp(-H_0\,\delta t)K - K\exp(-H_0\,\delta t)\exp - H_0 t_2(a, b)| \,.$$

Thus, by Duhamel's formula, if

(3.2.6) $$L = 2 \sup_{\substack{x,y \in \mathrm{supp}(K) \\ 0 \leqslant s \leqslant \delta t}} |\exp(-(\delta t - s)H_0)[K, H_0]\exp(-sH_0)(x, y)| \,,$$

(3.2.7) $$\left| \int d\mu_{N+1}\,\Delta K(x) \right| \leqslant ML(\delta t)^2 P_t(a, b).$$

Similarly

(3.2.8) $$\left| \int d\mu_{N+1}\,\Delta h(x) \right| \leqslant 2ML^2(\delta t)^2 P_t(a, b).$$

Thus the first order contribution to $S_{N,P+1} - S_{N,P}$ is of magnitude less than $C P_t(a, b)(\delta t)^2 M$, where $C = \max(L, 2L^2)$.

Similar arguments show that the contribution from the rth order term is not greater than $(1/(2^N)^2)(t^r C^r/r!)P_t(a, b)M$.

Thus

(3.2.9) $$|S_{N,P+1} - S_{N,P}| \leqslant (1/(2^N)^2)(\exp tC - 1)\, P_t(a, b)M,$$

and hence the sequence $I_1, I_2, \ldots$ tends to a limit as N tends to infinity. ■

COROLLARY 3.2.3. *$\int \mathrm{d}\mu\, [a, b, \alpha, \beta, \kappa, t]\, g$ is polynomial in α, β and κ.*

Proof. This result follows from the fact that $\lim_{N\to\infty} I_N$ exists for all values of α, β and κ. ■

3.3. A superspace Fevnman-Kac-Ito formula

This subsection contains a key theorem, which establishes a Feynman-Kac-Ito formula, that is, a path integral formula for the heat kernel of certain differential operators on the space of $L^{2'}$ functions on the superspace $B_L^{n,n}$. This formula will be used in the next section to prove the Gauss-Bonnet-Chern theorem.

THEOREM 3.3.1. *For $i = 1, \ldots, n$ and $\mu, \upsilon \in M_n$ let $h^{\upsilon}_{i\mu}$ and K^{υ}_{μ} be complex valued C^{∞} functions on R^n with compact support. Let g be a Riemannian metric on R^n equal to the Euclidean metric outside a compact region. Then, if H is the closure of the differential operator on $C^{\infty\prime}(B_L^{n,n}, C_L)$ defined by*

(3.3.1) $$H = \partial_i g^{ij} \partial_j + h^{\upsilon}_{i\mu}\theta^{\mu}\delta_{\upsilon}\partial_i + K^{\rho}_{\mu}\theta^{\mu}\delta_{\upsilon},$$

$$\exp - Ht(a, b, \alpha, \beta)$$

$$= \int \mathrm{d}^n\kappa \int \mathrm{d}\mu\, [a, b, \alpha, \beta, \kappa]$$

(3.3.2) $$\times \left\{\exp - \left[\int_0^t h^{\upsilon}_{i\mu}(x(s))\theta^{\mu}(s)\rho_{\upsilon}(s) g_{ij}(x(s))\, \mathrm{d}x^j(s)\right.\right.$$

$$\left. + \int_0^t (K^{\upsilon}_{\mu}(x(s)) - \partial_i h^i\mu\upsilon(x(s))\theta^{\mu}(s)\rho_{\upsilon}(u)\right\} \mathrm{d}s.$$

Proof. (This proof is modelled on a similar result for purely commuting variables given by Simon [24]). First a lemma is required.

LEMMA 3.3.2. *As in definition 3.2.1, for $s > 0$ let $P_s(x, y)$ be the integral kernel of $\exp - sH_0$ and let*

$$T_s : R_L^{2n,3n} \to C_L$$

with $T_s(x, y, \theta, \phi, \kappa)$

$$= P_s(x, y) \exp - i\kappa_i(\theta^i - \phi^i)$$

(3.3.3)
$$\times \exp\left\{-\frac{1}{2}\,(h^i(x, \theta, \kappa)g_{ij}(x) + h^i(y, \theta, \kappa)g_{ij}(y))\,(x^j - y^j)\right.$$
$$\left.\times \exp - s\; U(x, \theta, \kappa)\right\}$$

where $U(x, \theta, \kappa) = K(x, \theta, \kappa) - \partial_i h^i(x, \theta, \kappa)$

$$K(x, \theta, \kappa) = K_\mu^\nu(x)\theta^\mu \kappa_\nu$$

and $h^i(x, \theta, \kappa) = h_\mu^{i\nu}(x)\theta^\mu \kappa_\nu$.

Then (a) $\int \mathrm{d}^n\kappa\; T_s(x, y, \theta, \phi, \kappa)$ *is the integral kernel of a bounded operator* T_s *on* $L^{2'}(B_L^{n,n}, R)$, *and (b) for all f in the domain of H*

$$\lim_{s\to 0}\,(\mathrm{d}/\mathrm{d}s(T_s f)) = -Hf$$

Proof of lemma. Part (a) follows from the fact that

$$|\,T_s(x, y, \theta, \phi, \kappa)\,| \leqslant P_s(x, y)L$$

for some positive L (since $h_\mu^{i\nu}$ and K_μ^ν have compact support), together with the fact that $T_s(x, y, \theta, \phi, \kappa)$ is polynomial in θ, ϕ and κ.

(b) $\mathrm{d}/\mathrm{d}s\,(T_s f)$

$$= \int \mathrm{d}^n k\; \mathrm{d}^n y\; \mathrm{d}^n\phi\; P_s(x, y)\exp - i\kappa_i(\theta^i - \phi^i)$$
$$\times (-H_{oy}\left[\exp\left(-\frac{1}{2}\,(h^i(x, \theta, \kappa)g_{ij}(x) +\right.\right.$$
$$+ h^i(y, \theta, \kappa)g_{ij}(y))(x^i - y^i) - sT(x, \theta, \kappa))f(x, y)$$
$$- U(x, \theta, \kappa)\exp\left\{-\frac{1}{2}\,(h^i(x, \theta, \kappa)g_{ij}(x) +\right.$$
$$\left.+ h^i(y, \theta, \kappa)g_{ij}(y))\,(x^j - y^j) - sU(x, \theta, \phi)\right\}f(x, \phi)$$

■

$$= \int d^n k \, d^n y \, d^n \phi \, P_s(x, y) \exp - i\kappa_i(\theta^i - \phi^i)$$

(3.3.4)
$$\times \Big\{ - H_0 f(y, \phi) + (\partial_i g^{ij}) \Big[\frac{1}{2} (h^l(x, \theta, \kappa) g_{lj}(x) + h^l(y, \theta, \kappa) g_{lj}(y)) f(y, \phi) \Big]$$

$$+ 2 g^{ij} \partial_{iy} \Big[\frac{1}{2} (h^l(x, \theta, \kappa) g_{lj}(x) + h^l(y, \theta, \kappa) g_{lj}(y)) f(y, \phi) \Big]$$

$$- U(x, \theta, \kappa) f(x, \theta) + S_i(x, y, \theta, \phi, \kappa)(x^i - y^i) \Big\}$$

$$\times \exp \Big\{ - \frac{1}{2} (h^i(x, \theta, \kappa) g_{ij}(x) + h^i(y, \theta, \kappa) g_{ij}(y))(x^j - y^j) - sU(x, \theta, \phi) \Big\},$$

where $s_i(x, y, \theta, \phi, \kappa)$ is a smooth compact support function. Hence, since $P_s(x, y) \to \delta(x, y)$ as $s \downarrow 0$,

$$\lim_{s \to 0} d/ds \, (T_s f) = - Hf.$$

Proof of theorem 3.3.1. Let $S_t = \exp(-Ht)$. Then

$$\lim_{N \to \infty} || N(S_{t/N} - T_{t/N}) || = 0$$

by the lemma above.

Now $\lim_{N \to \infty} (T_{t/N})^N (a, b, \alpha, \beta)$ exists (by theorem 3.2.2 and corollary 3.2.3). Hence there must exist a positive constant C such that

$$|| (T_{t/N})^N || < C \text{ for all } N.$$

Thus

$$|| (T_{t/N})^N - S_t ||$$

$$= || \sum_{r=0}^{N-1} (T_{t/N})^r (T_{t/N} - S_{t/N})(S_{t/N})^{N-r-1} ||$$

$$< NC || T_{t/N} - S_{t/N} ||.$$

Thus $s\text{-lim} \, (T_{t/N})^N = \exp(-tH)$. ■

4. THE PROOF OF THE GAUSS-BONNET-CHERN THEOREM

In this section the local Gauss-Bonnet-Chern theorem is proved by using the superspace Feynman-Kac-Ito formula (Theorem 3.3.1) to evaluate the super kernel of $\exp(-\Delta'' t)$, Δ'' being the superspace version of the localised Laplacian.

THEOREM 4.1. *Let M be a compact orientable Riemannian manifold, with metric g and even dimension $n = 2k$, and let Δ denote the Laplacian on forms on M. Then at each point x of M*

$$\lim_{t \to 0} \operatorname{str} \exp(-\Delta t)(x, x) = E(x) \tag{4.1}$$

where, as in equation (1.3), E is the scalar function

$$E = (1/k!)(-1)^k (8\pi)^{-k}$$

$$\times \sum_{p,q \in \text{perm } 1,\ldots,n} (-1)^p (-1)^q R^{p(1)p(2)}_{q(1)q(2)} \ldots R^{p(n-1)p(n)}_{q(n-1)q(n)} .$$

Also the Euler number χ of M is given by the formula

$$\chi = \int_M E \, \mathrm{d}V. \tag{4.2}$$

((4.2) is the Gauss-Bonnet-Chern theorem, which follows immediately from (4.1) together with McKean and Singer's formula (1.4) [6]. *(4.1) is a stronger, local result originally conjectured by McKean and Singer, and first proved by Patodi* [7]*).*

Proof. (The idea of the proof is to rescale the odd parameters which occur in the superspace calculation of the supertrace, and to use the fact that $t^{n/2} P_t(0, 0)$ remains finite as t tends to zero, to isolate the contribution to the supertrace which survives in this limit from the contribution which dies away; the use of the heat kernel of the scalar Laplacian, rather than the more geometric Bochner Laplacian, makes this separation very simple).

With the notation of section 2.3,

$$\operatorname{str} \exp(-\Delta t)(x, x) = \mathrm{d}^n \alpha \exp(-\Delta'' t)(0, 0, \alpha, \alpha).$$

Now Δ'' has the form

$$\Delta'' = -\partial_i g^{ij'} \partial_j + S^{il}_k \theta^k \delta_l \partial_i + T^l_k \theta^k \delta_l + U^{mn}_{kl} \theta^k \theta^l \delta_m \delta_n,$$

where $U^{mn}_{kl} = \frac{1}{2} R'^{mn}_{kl} - g'_{ij} \Gamma'^m_{ik} \Gamma'^{ln}_j$, qnd S^{il}_k, T^l_k (and of course U^{mn}_{kl}) are smooth compact support functions on R^n. (Γ'^k_{ij} and R'^{kl}_{ij} are the connection and curvature coefficients corresponding to the metric g'. The precise form of S_k^{il} and T_k^l in terms of curvature and connection coefficients will not be important). Thus, using Theorem 3.3.1, with $H = \Delta''$,

$$\text{str}\exp(-tD)(x,x) = \int d^n\kappa\, d^n\alpha\, d\mu\,[0,0,\alpha,\alpha,\kappa,t]$$

$$\times \exp -\left\{\int_0^t S_{ik}^l(x(s))\theta^k(s)\rho_l(s)\,dx^i \right. \tag{4.3}$$

$$\left. + \int_0^t [T_k^l(x(s))\theta^k(s)\rho_l(s) + U_{kl}^{mm}(x(s))\theta^k(s)\theta^l(s)\,\rho_m(s)\,\rho_n(s)]\,ds\right\}$$

Inspection of the fermionic part of the measures $d\mu\,[0, 0, \alpha, \alpha, \kappa, t]$ and $d\mu\,[0, 0, 0, 0, 0, t]$ shows that this equation may be rewritten as

$$\text{str}\exp(-\Delta t)(x,x) = \int d^n\kappa\, d^n\alpha\, d\mu\,[0,0,0,0,0,t]$$

$$\times \exp -\left\{\int_0^t S_{ik}^l(x(s))(\theta^k(s)+\alpha^k)(\rho_l(s)+\delta_l)\,dx^i \right. \tag{4.4}$$

$$+ \int_0^t [T_k^l(x(s))(\theta^k(s)+\alpha^k)(\rho_l(s)+\kappa_l)$$

$$\left. + U_{kl}^{mm}(x(s))(\theta^k(s)+\alpha^k)(\theta^l(s)+\alpha^l)\,(\rho_m(s)+\kappa_m)(\rho_n(s)+\kappa_n]\,ds\right\}.$$

Setting $\alpha = t^{-1/4}\alpha'$, $\kappa = t^{-1/4}\kappa'$ (and hence $d\alpha = t^{1/4}\,d\alpha'$ and $d\kappa = t^{1/4}\,d\kappa'$)gives

$$\text{str}\exp(-\Delta t)(x,x) = \int d^n\kappa'\, d^n\alpha'\, t^{n/2}\, d\mu\,[0,0,0,0,0,t]$$

$$\times \exp -\left\{\int_0^t t^{-1}\, U_{kl}^{mn}(0)\,\alpha^k\alpha^l\kappa_m\kappa_n\,ds \right. \tag{4.5}$$

$$+\sum_{a=0}^{4}\left[\int_0^t t^{-a/4} f_{ai}(x(s),\theta(s),\rho(s))\,\mathrm{d}x^i + \right.$$

$$\left. + \int_0^t t^{-a/4} g_a(x(s),\theta(s),\rho(s))\,\mathrm{d}s\right\},$$

where the f_{ai} and g_a are of compact support, f_{4i} is zero, and (for $a = 1, 2, 3$) $f_{ai}(0,\theta,\rho) = 0$ for all (θ,ρ) in $R_L^{0,2n}$; and also, for $a = 1, 2, 3, 4, g_a(0,\theta,\rho)$ is zero for all (θ, ρ) in $R_L^{0,2n}$. (These properties of g_{ai} and f_a follow from the fact that normal coordinates are being used on M at x). Now consider

$$I(t) =_{\mathrm{def}} \int t^{n/2}\,\mathrm{d}\mu\,[0,0,0,0,0,t]$$

(4.6) $$\times\left\{\exp\sum_{a=0}^{4}\left[\int_0^t t^{-a/4} f_{ai}(x(s),\theta(s),\rho(s))\,\mathrm{d}x^i + \right.\right.$$

$$\left.\left. + \int_0^t t^{-a/4} g_a(x(s),\theta(s),\rho(s)\;\;\mathrm{d}s\right] - 1\right\}.$$

For $N = 1, 2, \ldots$, let $I_N(t)$ denote successive approximations to $I(t)$ according to the definition 3.1.2. Now it is proved in [12] that

(4.7) $$\lim_{t\to 0} t^{n/2} P_t(0,0) = (4\pi)^{-n/2}.$$

Hence $t^{-n/2}P_t(0, 0)$ is bounded for $0 < t < T$ (where T is some fixed positive number). Thus the proof of theorem 3.2.2 shows that $I_N(t)$ tends to $I(t)$ uniformly in t for $0 < t < T$. Hence

(4.8) $$\lim_{t\to 0} I(t) = \lim_{N\to\infty}\lim_{t\to 0} I_N(t)$$

if the right hand side of this equation exists. However arguments similar to those used in the proof of theorem 3.2.2 show that $\lim_{t\to 0} I_N(t) = 0$ for every N. Hence

(4.9) $$\lim_{t\to 0} I(t) = 0,$$

thus

$$\lim_{t\to 0} \operatorname{str} \exp(-\Delta t)(x, x)$$

$$= \lim_{t\to 0} \int d^n\kappa\, d^n\alpha\, d\mu\, [0, 0, 0, 0, 0, t] \exp - U_{k1}^{mn}(0)\alpha^k\alpha^1\kappa_m\kappa_n$$

$$= (4\pi)^{-k} \int d^n\kappa\, d^n\alpha \exp - \frac{1}{2} R_{k1}^{mn}(0)\alpha^k\alpha^1\kappa_m\kappa_n \tag{4.10}$$

$$= (1/k!)(-1)^k(B\pi)^{-k}$$

$$\times \sum_{p,q\in \operatorname{perm}\{1,\ldots,n\}} (-1)^p(-1)^q R_{q(1)q(2)}^{p(1)p(2)}(x) \ldots R_{q(n-1)q(n)}^{p(n-1)p(n)}(x),$$

as required. ■

CONCLUSION

This paper shows that fermionic path integration can be used to derive analytic results. The integrals developed here are the minimum necessary for the proof of the Gauss-Bonnet-Chern theorem, but various extensions should be possible. For instance, the Feynman-Kac-Ito formula (Theorem 3.3.1) should remain valid under rather less stringent conditions on the function $h_{i\mu}^{\nu}$ and K_{μ}^{ν}. Also more general subdivisions of the time interval in the definition of the integrand for the Feynman-Kac-Ito formula should be possible.

Work is in progress to extend the methods of this paper to the twisted Dirac operator, and hence to obtain a proof of the Atiyah-Singer intex theorem for all the classical complexes.

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